Convergent Twist Deformations

An analytic framework for Drinfeld Twists

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Joint work with Chiara Esposito and Stefan Waldmann

Setting the Stage

$$\varrho \colon \mathfrak{g} \longrightarrow \mathrm{L}(V)$$

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Lie algebra

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Representation

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- Lie algebra: $\mathfrak{g}$
- Locally convex space: V
- Continuous linear endomorphisms: $\mathrm{L}(V)$
- Representation: ϱ

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Act I: Algebra

Universal Enveloping algebra

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and

$$\epsilon(x) = 0 \qquad \text{as well as} \qquad S(x) = -x$$

for all $x \in \mathfrak{g} = \mathrm{U}^1(\mathfrak{g})$.

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where $\Delta(x)$ acts on $\mathfrak{A} \otimes \mathfrak{A}$ diagonally and $\varrho(1) = \mathrm{id}_{\mathfrak{A}}$.

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Drinfeld Twist F is

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Get action of F, denoted by $F \triangleright \cdot$.

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$$a \star_F b := \mu\big(F \triangleright (a \otimes_{\mathbb{C}[\![\hbar]\!]} b)\big) \qquad \text{for all } a, b \in \mathfrak{A}[\![\hbar]\!]$$

endows $\mathfrak{A}[\![\hbar]\!]$ *with the structure of a associative algebra with unit* η.

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Upshot: One twist works for every representation!

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- Equivariant $\mathfrak{g}$-triple if

$$\xi \triangleright \mu(v \otimes w) = \mu(\xi \triangleright v \otimes w) + \mu(v \otimes \xi \triangleright w)$$

for all $\xi \in \mathfrak{g}$, $v \in V$ and $w \in W$.

Definition (Morphisms)

A morphism between a $\mathfrak{g}$-triple and a $\widetilde{\mathfrak{g}}$-triple

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for all $v \in V$ and $w \in W$.

If (C, Δ, ϵ) is coassociative counital coalgebra and

$$\varrho \colon \mathfrak{g} \longrightarrow \mathrm{CoDer}(C)$$

is a Lie algebra representation by co-derivations of C, then setting

$$\Delta_F := \varrho(F) \circ \Delta$$

yields a coassociative coproduct on $C[\![\hbar]\!]$ with counit η.

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constitutes a Drinfeld twist.

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The only term containing 1 is $1 \otimes 1$, which occurs precisely once. Moreover,

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If $r = r_1 \otimes r_2$, then $(\Delta \otimes \mathrm{id})(r) = 1 \otimes r_1 \otimes r_2 + r_1 \otimes 1 \otimes r_2$. □

Giaquinto-Zhang: $ax + b$

Let $\mathfrak{g} = \operatorname{span}\{H, E\}$ with $[H, E] = E$. We write

$$H_{k\uparrow} := H \cdot (H+1) \cdots (H+k-1)$$

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with

$$F_n := \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot (E^{n-k} \cdot H_{k\uparrow}) \otimes (E^k \cdot H_{(n-k)\uparrow}).$$

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Problem: Translation fairly inexplicit.

Act II: Representation Theory

Naive Lie correspondence: $\mathrm{L}(V)$ exponentiates into

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Extremely badly behaved group! Nevertheless,

$$\pi\big(\exp \xi\big) v := \exp\big(\varrho(\xi)\big) v = \sum_{n=0}^{\infty} \frac{\varrho(\xi)^n v}{n!}$$

for all $v \in V$ and $\xi \in \mathfrak{g}$ such that the series converges.

Entire vectors

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for all $r \geq 0$, continuous seminorms $\mathrm{q} \in \mathrm{cs}(V)$.

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for all $r \geq 0$, continuous seminorms $\mathrm{q} \in \mathrm{cs}(V)$. Write $\mathscr{E}(\varrho)$ for the space of entire vectors and endow it with locally convex topology induced by $\mathrm{p}_{r,\mathrm{q}}$.

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Theorem (Lie-Taylor formula)

Get group representation $\pi \colon G \longrightarrow \mathrm{GL}(\mathscr{E}(\varrho))$ *via*

$$\pi(\exp \xi) v := \exp(\varrho(\xi)) v.$$

Consider $\mathscr{L} \colon \mathfrak{g} \longrightarrow \mathrm{L}(\mathscr{C}^\infty(G))$ as left-invariant differential operators.

Universal Example

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Theorem (H., 2025)

Let G be a connected Lie group. Then

$$\mathscr{E}(\mathscr{L}) = \mathscr{H}(G_{\mathbb{C}})$$

as Fréchet algebras, where $G_{\mathbb{C}}$ denotes the universal complexification of G.

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Proposition (Esposito, H., Waldmann 2025)

Let $\pi \colon G \longrightarrow \mathrm{GL}(V)$ be a Lie group representation of a connected Lie group with corresponding $\varrho = T_{\mathrm{e}}\pi \colon \mathfrak{g} \longrightarrow \mathrm{L}(V)$.

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$$\pi_{v,\varphi}\colon G \longrightarrow \mathbb{C}, \quad \pi_{v,\varphi}(g) := \varphi\big(\pi(g)v\big).$$

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Definition (Analytic vectors)

A vector $v \in V$ is called analytic with radius of convergence $r_0 > 0$ if

$$\mathrm{p}_{r,\mathrm{q}}(v) := \sum_{n=0}^{\infty} \frac{r^n}{n!} \cdot \sup_{\xi_1,\dots,\xi_n \in \mathbb{B}} \mathrm{q}\big(\xi_1 \cdots \xi_n \triangleright v\big) < \infty$$

for all $0 \leq r < r_0$ and continuous seminorms $\mathrm{q} \in \mathrm{cs}(V)$.

Analytic vectors

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A vector $v \in V$ is called analytic with radius of convergence $r_0 > 0$ and of order $R \geq 0$ if

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We have

$$\mathscr{E}_R(\varrho) := \varprojlim_{r_0 > 0} \mathscr{A}_{R,r_0}(\varrho).$$

Lemma

If V is Hausdorff, then so are $\mathscr{E}_R(\varrho)$, $\mathscr{A}_{R,r_0}(\varrho)$ and $\mathscr{A}_R(\varrho)$.

First properties

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Lemma

The assignment $\mathscr{A}_{R,r}\colon \mathrm{Rep} \longrightarrow \mathrm{Rep}$,

$$(\varrho\colon \mathfrak{g} \longrightarrow \mathrm{L}(V)) \mapsto (\varrho\colon \mathfrak{g} \longrightarrow \mathrm{L}(\mathscr{A}_{R,r}(\varrho)))$$

acting as restriction on continuous intertwiners constitutes a covariant functor.

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Example

Let $\mathfrak{g}$ be abelian. Then the space $\mathscr{E}_R(\mathscr{L})$ is the space of entire functions on $\mathfrak{g}_{\mathbb{C}}$ of complex analytic order at most $1/R$.

Act III: Convergent Universal Deformations
Abstract Nonsense

Proposition

Let $\mu\colon V \otimes W \longrightarrow X$ *be a continuous equivariant* $\mathfrak{g}$*-triple.*

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Let $\mu \colon V \otimes W \longrightarrow X$ be a continuous equivariant $\mathfrak{g}$-triple. Then so is

$$\mu \colon \mathcal{E}_R(\varrho_V) \otimes \mathcal{E}_R(\varrho_W) \longrightarrow \mathcal{E}_R(\varrho_X).$$

This yields an endofunctor.

Let $F = \sum_{n=0}^{\infty} \hbar^n/n! \cdot F_n$ be a twist and $R \geq 0$ with:

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Equicontinuity Condition

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$$\left(\mathrm{p}^{(R)}_{r,\mathrm{q}_V} \otimes \mathrm{p}^{(R)}_{r,\mathrm{q}_W}\right)\left(\frac{\hbar^n}{n!} F_n \triangleright (v \otimes w)\right) \leq$$

Deformation of Entire Vectors

Let $F = \sum_{n=0}^{\infty} \hbar^n / n! \cdot F_n$ be a twist and $R \geq 0$ with:

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$$\left(\mathrm{p}^{(R)}_{r,\mathrm{q}_V} \otimes \mathrm{p}^{(R)}_{r,\mathrm{q}_W}\right)\left(\frac{\hbar^n}{n!} F_n \triangleright (v \otimes w)\right) \leq C \cdot \mathrm{p}^{(R)}_{tr,\mathrm{q}'_V}(v) \cdot \mathrm{p}^{(R)}_{tr,\mathrm{q}'_W}(w)$$

for all $v \in \mathscr{E}_R(\varrho_V)$, $w \in \mathscr{E}_R(\varrho_W)$, $\hbar \in K$ and $n \in \mathbb{N}_0$.

Theorem (Esposito, H., Waldmann, 2026)

Let $\mu\colon V \otimes W \longrightarrow X$ be a continuous equivariant $\mathfrak{g}$-triple and assume the equicontinuity condition.

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Let $\mu\colon V \otimes W \longrightarrow X$ be a continuous equivariant $\mathfrak{g}$-triple and assume the equicontinuity condition. Then

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$$\mathcal{D}_\hbar\colon \mathrm{emTriple}_{\mathfrak{g}} \longrightarrow \mathrm{cTriple}_{\mathfrak{g}}.$$

Universal Deformation

Theorem (Esposito, H., Waldmann, 2026)

Let $\mu\colon V \otimes W \longrightarrow X$ *be a continuous equivariant* $\mathfrak{g}$*-triple and assume the equicontinuity condition. Then*

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Similar statement for separate continuity and analytic vectors.

We call an inductive system

$$V = \varinjlim_{\alpha \in J} V_\alpha$$

fully reduced if the connecting maps are set inclusions.

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fully reduced if the connecting maps are set inclusions. In this case,

$$V = \bigcup_{\alpha \in J} V_\alpha$$

as vector spaces.

Everyone is an inductive limit!

Definition (Inductive)

A $\mathfrak{g}$-triple is called inductive if $V = \varinjlim V_\alpha$, $W = \varinjlim W_\beta$ and $X = \varinjlim X_\gamma$ are fully reduced inductive systems such that:

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- Each Z_α is ϱ_Z-invariant for $Z = V, W, X$.

TUDelft

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Lemma

This implies the separate continuity of $\mu \colon V \otimes W \longrightarrow X$.

Guiding Example

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  for compact sets $K \subseteq M$ are.
- Representations by differential operators leave $\mathscr{C}^\infty_K(M)$ invariant!

Definition (Analytic Vectors II)

If V is a fully reduced inductive system, then we define its space of analytic vectors as

$$\mathscr{A}_R(\varrho) := \varinjlim_{\alpha\in J, r_0>0} \mathscr{A}_{R,r_0}(\varrho_\alpha),$$

where $\varrho_\alpha := \varrho\big|_{V_\alpha}$.

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where $\varrho_\alpha := \varrho\big|_{V_\alpha}$.

If the limit is strict, then this agrees with both iterative ways of taking inductive limits.

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Act IV: Convergent Universal Deformations
Concrete Considerations

Proposition (Esposito, H., Waldmann, 2026)

Let $\mathfrak{g}$ be abelian and $R = 1/2$.

Equicontinuity

Proposition (Esposito, H., Waldmann, 2026)

Let $\mathfrak{g}$ be abelian and $R = 1/2$. Then the abelian twists $\exp(\hbar r)$ with $r \in \mathfrak{g} \otimes \mathfrak{g}$ fulfil the equicontinuity condition.

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Proposition (Esposito, H., Waldmann, 2026)

Let $\mathfrak{g} = \mathrm{span}\{H, E\}$ with $[H, E] = E$ and $R = 1$. Then the Giaquinto-Zhang twist fulfils the equicontinuity condition.

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More complicated Giaquinto-Zhang twist: also $R = 1$ for all dimensions.

Let

$$\varrho(H) := -z\frac{\mathrm{d}}{\mathrm{d}z} \qquad \text{and} \qquad \varrho(E) := \frac{\mathrm{d}}{\mathrm{d}z}$$

acting on $V = \mathscr{H}(\mathbb{C})$, endowed with the compact-open topology.

A concrete representation

Let

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Proposition (Esposito, H., Waldmann, 2026)

We have $\mathscr{E}_0(\varrho) = \mathscr{H}(\mathbb{C})$ and $\mathscr{A}_1(\varrho) = \mathscr{H}_1(\mathbb{C})$ as locally convex spaces, where $\mathscr{H}_1(\mathbb{C})$ denotes the space of entire functions of complex analytic order at most one.

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Proposition (H., Roth, Waldmann, 2022)

Matrix elements of finite dimensional representations are within $\mathscr{A}_1(\mathscr{L})$.

Act V: Reaching for the Stars

Where do we go...?

Understand the following in the analytic framework:

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- Equivalence of twists: Topology?

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TUDelft

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- Quantization of symmetries to Quantum Groups.
- Locally convex Lie algebras.

Last slide.

This is the last slide. Take a look at the piece of paper or screen your notes rest on. It is probably empty. On the off-chance that it is not, the time has come to make yourself heard!